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RESEARCH ARTICLE

Short-term relative arbitrage in volatility-stabilized markets

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Abstract We answer in the affirmative the following open question posed in Fernholz and Karatzas (Ann Finance 1:149–177, 2005): do there exist relative arbitrage opportunities over arbitrarily short time horizons in the context of certain volatility-stabilized market models?

Keywords Portfolios · Portfolio generating functions · Continuous semimartingales

JEL Classification G10

1 Introduction

Several recent results in stochastic portfolio theory have been concerned with the existence of relative arbitrage in equity markets. Roughly speaking, relative arbitrage over a given time horizon occurs if there exists a pair of all-long portfolios of equal initial value such that the first portfolio is guaranteed not to underperform the second, and such that the probability of outperformance is nonzero. We shall consider the special case of strong relative arbitrage, wherein the first portfolio outperforms the second with probability one.

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In Fernholz et al. (2005), it is shown that strong relative arbitrage exists over arbitrarily short time horizons in the context of equity market models which resemble actual equity markets. The key property of these so-called weakly diverse markets, so far as relative arbitrage is concerned, is that the relative volatility (with respect to the market) of the stock of largest capitalization admits an a.s. positive lower bound.

On the other hand, two of the above authors have developed a market model with extreme volatility at the low end of the capitalization scale, namely the *volatility-stabilized model* of Fernholz and Karatzas (2005). In the aforementioned article, it is shown that the extreme volatilities enjoyed by the small-cap stocks in this model lead to strong relative arbitrage on time horizons greater than a fixed constant which depends on the number of stocks in the market. The article poses the following question: does relative arbitrage exist in this market model on arbitrarily short time horizons?

We answer this question in the affirmative; in fact, the same result holds for any market whose smallest-cap stock possesses sufficient relative volatility. Section 2 establishes the details of the market model and provides the formal definition of strong relative arbitrage opportunities. Section 3 contains a proof of our main result. Finally, having resolved one open question, we retaliate with another in Sect. 4.

2 Preliminaries

We refer the reader to Sect. 2 of Fernholz and Karatzas (2005) for an explanation of the equity market model under consideration. In particular, equation (2.3) of that paper gives the behavior of the log capitalization processes:

$$d\,(\log X_i(t)) = \gamma_i(t)\,dt + \sum_{\nu=1}^{n} \sigma_{i\nu}(t)\,dW_\nu(t), \quad i = 1, \ldots, n. \tag{2.1}$$

A special case is the so-called volatility-stabilized market given by

$$d\,(\log X_i(t)) = \frac{\alpha}{2\mu_i}\,dt + \frac{1}{\sqrt{\mu_i(t)}}\,dW_i(t), \quad i = 1, \ldots, n; \tag{2.2}$$

here $\mu_i(\cdot) = X_i(\cdot)/\sum_{j=1}^{n} X_j(\cdot)$ is the market weight of the ith security and $\alpha \geq 0$ is constant. In this model, the relative variances $\tau_{ii}^{\mu}(\cdot)$ [see (2.22) of Fernholz and Karatzas (2005) for the general definition] are given by

$$\tau_{ii}(t) := \tau_{ii}^{\mu}(t) = \frac{1}{\mu_i(t)} - 1. \tag{2.3}$$

Given a fixed time horizon $[t_0, T]$, we shall say that a *strong relative arbitrage opportunity* over $[t_0, T]$ is a pair of portfolios $(\pi(\cdot), \rho(\cdot))$ such that there exists a constant $q = q_{\pi,\rho,t_0,T} > 0$ satisfying

$$P\left(\frac{Z^{\pi}(t)}{Z^{\rho}(t)} \geq q, \text{ for all } t_0 \leq t \leq T\right) = 1, \tag{2.4}$$

$$P[Z^{\pi}(T) > Z^{\rho}(T)] = 1 \tag{2.5}$$

whenever $Z^{\pi}(t_0) = Z^{\rho}(t_0) = z > 0$. Here $Z^{\pi}(\cdot)$ and $Z^{\rho}(\cdot)$ are the portfolio value processes corresponding to $\pi(\cdot)$ and $\rho(\cdot)$, respectively, as in (2.5) of Fernholz and Karatzas (2005). [Note that our condition (2.5) above is stronger than the corresponding condition (2.14) in Fernholz and Karatzas (2005), hence the moniker "strong relative arbitrage."] Fernholz and Karatzas show that a strong relative arbitrage opportunity exists in the market model of (2.2) above over the time horizon $[0, T]$ for any $T > T_*$, where

$$T_* := \frac{2S(\mu(t_0))}{n-1}; \tag{2.6}$$

here $S(\cdot)$ is the *entropy function* on the simplex $\Delta^n := \{x \in \mathbf{R}^n : \sum_{i=1}^n x_i = 1, x_i > 0$ for all $i\}$ defined by

$$S(x) := -\sum_{i=1}^{n} x_i \log(x_i), \qquad x = (x_1, \ldots, x_n) \in \Delta^n. \tag{2.7}$$

Our goal is to show that a strong relative arbitrage opportunity exists over $[0, T]$ for *any* $T > 0$. In fact, we shall establish this fact for any market of the form (2.1) satisfying the condition

$$\tau_{m(t)m(t)}(t) \geq \frac{C}{\mu_{m(t)}(t)} \qquad \text{for all } t \geq 0 \tag{2.8}$$

almost surely, where $C > 0$ is constant and $m(t)$ denotes the index of the stock of minimum weight at time t, so that $\mu_{m(\cdot)} = \min\{\mu_1(\cdot), \ldots, \mu_n(\cdot)\}$. (Ties can be resolved by selecting the smallest appropriate index.) In other words, (2.8) states that the relative variance of the stock of smallest weight is at least of the order of the reciprocal of its weight. Equation (2.3) shows that the above condition is satisfied (with $C = 1/2$) in the volatility-stabilized model of (2.2) for $n \geq 2$.

3 The main result

In this section, we prove the following:

Proposition 1 *For any* $T > 0$, *a strong relative arbitrage opportunity exists over the time horizon* $[0, T]$ *in any market model of the form* (2.1) *satisfying the condition* (2.8).

Proof We use the theory of portfolio generating functions, as described in Sect. 3.1 of Fernholz (2002).

For some constant $c > 1$ to be determined, define a function f on $[0, 1]$ by setting

$$f(y) = \begin{cases} \Gamma(c+1, -\log y) = \int\limits_{-\log(y)}^{\infty} e^{-r} r^c \, dr, & \text{if } 0 < y \le 1, \\ 0, & \text{if } y = 0. \end{cases}$$

[Here $\Gamma(\cdot, \cdot)$ is the standard incomplete gamma function.] It is straightforward to check that f is continuous at 0, $f(1) < \infty$, and that

$$f'(y) = (-\log y)^c, \qquad f''(y) = -\frac{c(-\log y)^{c-1}}{y} \qquad \text{on } (0, 1), \tag{3.1}$$

so that f is C^2, nonnegative, bounded, increasing and concave; furthermore, the function $y \mapsto yf'(y)$ is bounded on $(0, 1)$. This allows us to consider the portfolio π generated by $\mathbf{S}$, where

$$\mathbf{S}(x_1, \dots, x_n) := \sum_{i=1}^{n} f(x_i). \tag{3.2}$$

Assuming that $Z^\pi(0) = Z^\mu(0)$, we then have from (3.1.1) and (3.1.8) of Fernholz (2002) that

$$\log\left(\frac{Z^\pi(t)}{Z^\mu(t)}\right) = \log \mathbf{S}(\mu(t)) - \log \mathbf{S}(\mu(0)) + \int\limits_0^t \frac{1}{2\mathbf{S}(\mu(s))} \sum_{i=1}^{n} -f''(\mu_i(s))\tau_{ii}(s)\mu_i^2(s)\, ds, \tag{3.3}$$

almost surely. Now, for $x \in \Delta^n$, put $x_{(n)} := \min\{x_1, \dots, x_n\}$ and note that $0 < x_{(n)} \le 1/n$. In the Appendix, we shall prove the estimates

$$\mathbf{S}(x) = \sum_{i=1}^{n} f(x_i) \le f(x_{(n)}) + (n-1) f\left(\frac{1 - x_{(n)}}{n-1}\right) \le nf(1/n) \tag{3.4}$$

and

$$\mathbf{S}(x) \ge (n-1)f(x_{(n)}) + f(1-(n-1)x_{(n)}) \ge (n-1)f(0) + f(1) = f(1). \tag{3.5}$$

Setting $y_\cdot := \mu_{(n)}(\cdot)$ for convenience, we see that these estimates, (2.8), and (3.3) lead to

$$\begin{aligned}
\log\left(\frac{Z^\pi(t)}{Z^\mu(t)}\right) &\geq \log \mathbf{S}(\mu(t)) - \log \mathbf{S}(\mu(0)) + \int_0^t \frac{1}{2\mathbf{S}(\mu(s))}\left(-f''(y_s)\right)\frac{C}{y_s}(y_s^2)\,ds \\
&\geq \left[\log((n-1)f(y_t) + f(1-(n-1)y_t))\right] - \log(nf(1/n)) \\
&\quad + \int_0^t \frac{-Cy_s f''(y_s)}{2\left(f(y_s) + (n-1)f\left(\frac{1-y_s}{n-1}\right)\right)}\,ds \qquad \text{a.s.} \\
&=: S_1(y_t) - \log(nf(1/n)) + \int_0^t \Theta_1(y_s)\,ds.
\end{aligned} \tag{3.6}$$

In light of (3.5) and the nonnegativity of $f(\cdot)$ and $\Theta_1(\cdot)$, we see that the condition (2.4) is satisfied for the pair $(\pi(\cdot), \mu(\cdot))$ with $q = f(1)/(nf(1/n))$. We now claim that

$$\Theta_1(\cdot) \text{ is decreasing on } (0, 1/n). \tag{3.7}$$

Indeed, the numerator $-Crf''(r) = cC(-\log r)^{c-1}$ of $\Theta_1(r)$ is clearly decreasing in r. As for the denominator, we note that

$$\frac{1-r}{n-1} > r \qquad \text{for } r \in (0, 1/n);$$

now, since f is concave, f' is decreasing, so

$$f'(r) - f'\left(\frac{1-r}{n-1}\right) > 0 \qquad \text{for } r \in (0, 1/n).$$

The left-hand side of this inequality is one-half of the derivative of the denominator of $\Theta_1(r)$ (with respect to r). This shows that this denominator is increasing in r, hence $\Theta_1(\cdot)$ is indeed decreasing on $(0, 1/n)$.

Now fix $t_0 \geq 0$, and define a function $T_1(\cdot)$ on $[0, 1/n]$ by

$$\begin{aligned}
T_1(Y) : &= t_0 + \int_{1/n}^{Y} -\frac{S_1'(r)}{\Theta_1(r)}\,dr \\
&= t_0 + \int_{Y}^{1/n} \frac{(n-1)f'(r) - (n-1)f'(1-(n-1)r)}{(n-1)f(r) + f(1-(n-1)r)} \\
&\qquad \times \left(\frac{-Crf''(r)}{2\left(f(r) + (n-1)f\left(\frac{1-r}{n-1}\right)\right)}\right)^{-1} dr.
\end{aligned} \tag{3.8}$$

To see that $T_1(0)$ is well defined, note that (3.1) implies that

$$T_1(0) \le t_0 + \frac{(n-1)2nf(1/n)}{f(1)} \int_0^{1/n} \frac{f'(r)}{-Crf''(r)}\,dr \tag{3.9}$$

$$\le t_0 + \frac{2n(n-1)}{C} \int_0^{1/n} \frac{-\log r}{c}\,dr = t_0 + \frac{A_n}{c}, \tag{3.10}$$

where $A_n < \infty$ is independent of c. Here we have used the estimates (3.4) and (3.5), as well as the fact that $f'(1-(n-1)r) > 0$ for all r in $(0, 1/n)$, to obtain (3.9), while the first inequality in (3.10) follows from the simple observation that $f(1/n) \le f(1)$.

The function $T_1(\cdot)$ satisfies the differential equation $T_1'(Y) = -S_1'(Y)/\Theta_1(Y)$. Since $S_1'(Y) > 0$ and $\Theta_1(Y) > 0$ for all $Y \in (0, 1/n)$, we see that $T_1(\cdot)$ is decreasing. It therefore possesses an inverse $Y(\cdot)$, defined on the interval $[t_0, T_1(0)]$. Since $Y'(t) = 1/T_1'(Y(t))$, we have $S_1'(Y(t))Y'(t) + \Theta_1(Y(t)) = 0$. In light of the initial condition $Y(t_0) = T_1^{-1}(t_0) = 1/n$, we see that $Y(\cdot)$ satisfies the integral equation

$$S_1(Y(t)) + \int_{t_0}^{t} \Theta_1(Y(s))\,ds \equiv S_1(1/n) = \log(nf(1/n)), \qquad t \in [t_0, T_1(0)]. \tag{3.11}$$

We now set $c = 2A_n/T$, so that

$$\frac{A_n}{c} = T/2. \tag{3.12}$$

Set $t_0 = T/2$, let $Y(\cdot)$ be as in (3.8) above, and put

$$T_0 = \inf\{t \ge T/2 : y_t > Y(t)\}.$$

(Recall that $y_\cdot := \mu_{(n)}(\cdot)$.) Clearly T_0 is a stopping time, and we also claim that $T_0 \le T$ a.s.; indeed, since $y_{T_1(0)} > 0$ a.s. [for the function $T_1(\cdot)$ defined in (3.8) above], we must have $T_0 \le T_1(0)$. On the other hand, (3.10) and (3.12) show that $T_1(0) \le T$, so $T_0 \le T$ a.s., as claimed.

Define a portfolio $\tilde{\pi}(\cdot)$ by setting

$$\tilde{\pi}(t) = \begin{cases} \pi(t), & t < T_0 \\ \mu(t), & t \ge T_0. \end{cases}$$

Since the condition (2.4) is satisfied for the pair $(\pi(\cdot), \mu(\cdot))$ and $q = f(1)/(nf(1/n))$, it is clearly also satisfied for the pair $(\tilde{\pi}(\cdot), \mu(\cdot))$ with the same value of q. It remains to establish the condition (2.5) for the pair $(\tilde{\pi}(\cdot), \mu(\cdot))$.

We now return to the estimate (3.6). Using the facts that $y_t \le 1/n$ on $[0, T/2]$, $y_t \le Y(t)$ on $[T/2, T_0]$, and $\Theta_1(\cdot)$ is decreasing, as well as (3.11), we have

$$\begin{aligned}\log\left(\frac{Z^{\tilde{\pi}}(T)}{Z^{\mu}(T)}\right) &= \log\left(\frac{Z^{\pi}(T_0)}{Z^{\mu}(T_0)}\right)\\ &\ge S_1(y_{T_0}) - \log(nf(1/n)) + \int_0^{T/2} \Theta_1(y_s)\,ds + \int_{T/2}^{T_0} \Theta_1(y_s)\,ds\\ &\ge S_1(Y(T_0)) - \log(nf(1/n)) + \int_0^{T/2} \Theta_1(1/n)\,ds + \int_{t_0}^{T_0} \Theta_1(Y(s))\,ds\\ &= \int_0^{T/2} \Theta_1(1/n)\,ds = (T/2)\Theta_1(1/n), \qquad \text{a.s.}\end{aligned}$$

This establishes the desired relative strong arbitrage, since the quantity $(T/2)\Theta_1(1/n)$ is a positive constant (depending on T, C, c and n). □

Note that the precise form of the growth rate terms $\gamma_i(\cdot)$ of (2.1) does not affect the arbitrage. In other words, the relative arbitrage is driven purely by volatility considerations. Also, let us briefly examine the constants A_n and $(T/2)\Theta_1(1/n)$ which appear in the above proof. For fixed large n, we have

$$A_n := \frac{2n(n-1)}{C}\int_0^{1/n} (-\log r)\,dr = \frac{2(n-1)(\log n + 1)}{C}.$$

Since $T/2 = A_n/c$, we have the following expression for the minimal amount of arbitrage guaranteed by the portfolio $\tilde{\pi}$:

$$(T/2)\Theta_1(1/n) = \frac{A_n}{c}\,\frac{Cc(\log n)^{c-1}}{2n\Gamma(c+1, \log n)} = \frac{(n-1)(\log n+1)(\log n)^{4(n-1)(\log n+1)/CT}}{n\Gamma(1+4(n-1)(\log n+1)/CT, \log n)}.$$

The denominator grows at the order of $[1/T]!$, so the above quantity tends to 0 very quickly as $T \to 0^+$.

4 Further considerations

Propositions 3.1 and 3.8 of Fernholz and Karatzas (2005) state that strong relative arbitrage opportunities exist over long enough time horizons in any market satisfying the condition

$$\Gamma(t) \le \int_0^t \gamma_*^{\mu,p}(s)\,ds < \infty, \qquad \text{a.s.} \tag{4.1}$$

for some $p > 0$ and continuous, strictly increasing function $\Gamma : [0, \infty) \to [0, \infty)$ with $\Gamma(0) = 0$ and $\Gamma(\infty) = \infty$. In the above equation, $\gamma_*^{\mu,p}(\cdot)$ is the *generalized excess growth rate* of the market, which can be expressed as a weighted-average relative volatility of the market:

$$\gamma_*^{\mu,p}(t) = \frac{1}{2}\sum_{i=1}^{n}(\mu_i(t))^p \tau_{ii}(t).$$

The existence of a long-term strong relative arbitrage opportunity in any market satisfying (2.8) follows as a corollary, since in that case $\gamma_*^{\mu,1}(t)$ is a.s. greater than $C/2$ for all t.

On the other hand, the proof of Proposition 1 relies on the large relative volatility enjoyed by the smallest-weight stock. In general, even if the condition (4.1) is replaced by the stronger condition

$$\gamma_*^{\mu,p}(t) \geq C \qquad \text{for all } t, \text{ a.s.,} \tag{4.2}$$

we have been unable to demonstrate that short-term strong relative arbitrage must exist for $n \geq 3$. When $n = 2$, the general equation $\sum_{i=1}^{n} \tau_{ij}(t)\mu_j(t) = 0$ for all j and $t \geq 0$ (c.f. Lemma 1.2.2 of Fernholz 2002) implies that $\mu_1^2(t)\tau_{11}(t) = \mu_2^2(t)\tau_{22}(t)$. If we assume that

$$\gamma_*^{\mu,1}(t) = \frac{1}{2}(\mu_1(t)\tau_{11}(t) + \mu_2(t)\tau_{22}(t)) \geq C,$$

then assuming without loss of generality that the smallest-weight stock at time t has index 1, we have

$$\mu_1(t)\tau_{11}(t) \geq \frac{1}{2}\left(\mu_1(t) + \frac{\mu_1^2(t)}{\mu_2(t)}\right)\tau_{11}(t) = \frac{1}{2}(\mu_1(t)\tau_{11}(t) + \mu_2(t)\tau_{22}(t)) \geq C.$$

That is, the condition (4.2) implies (2.8), so we do have short-term strong relative arbitrage in this case. However, we still have the following open question:
Open question: do strong relative arbitrage opportunities exist over arbitrarily short time horizons in any market model satisfying the condition (4.2), or even (4.1), for $n \geq 3$?

5 Appendix: some properties of concave functions

We now prove equations (3.4) and (3.5) from Sect. 3 by using the following pair of lemmas relating to concave functions:

Lemma 1 *If f is concave on $[A, B]$ and $a_1, \ldots, a_m \in [A, B]$, then*

$$\sum_{i=1}^{m} f(a_i) \leq mf\left(\frac{1}{m}\sum_{i=1}^{m} a_i\right).$$

Lemma 2 *Suppose that f is concave on $[A, B]$ and $a_1, \dots, a_m \in [A, B]$ are chosen such that $a - (m-1)A \leq B$, where $a := \sum_{i=1}^m a_i$. Then*

$$\sum_{i=1}^{m} f(a_i) \geq (m-1)f(A) + f(a - (m-1)A). \tag{5.3}$$

To prove these lemmas, we first recall that a concave function on $[A, B]$ satisfies the inequality

$$\sum_{i=1}^{m} w_i f(a_i) \leq f\left(\sum_{i=1}^{m} w_i a_i\right) \tag{5.4}$$

whenever $a_1, \dots, a_m \in [A, B]$ and $w := (w_1, \dots, w_m) \in \Delta^m$. Note that (5.4) follows easily (by induction) from the more commonly-quoted property

$$(1-\lambda)f(a_1) + \lambda f(a_2) \leq f((1-\lambda)a_1 + \lambda a_2), \tag{5.5}$$

valid for any $0 \leq \lambda \leq 1$ and a_1, a_2 in the domain of the concave function f. In any case, Lemma 1 follows from (5.4) by setting $w_i = 1/m$ for all $i = 1, \dots, m$. As for Lemma 2, we set $\lambda_i = (a_i - A)/(a - mA)$. It is straightforward to check that $0 \leq \lambda_i \leq 1$ and that

$$(1-\lambda_i)A + \lambda_i(a - (m-1)A) = a_i.$$

It follows from (5.5), with λ replaced by λ_i, that

$$f(a_i) \geq (1-\lambda_i)f(A) + \lambda_i f(a - (m-1)A)$$

for any $i = 1, \dots, m$. Adding all m inequalities establishes (5.3), since $\sum_{i=1}^m \lambda_i = 1$. □

In the case where $x := (x_1, \dots, x_n)$ lies in Δ^n, and $x_{(n)} := \min\{x_1, \dots, x_n\}$, we may as well suppose that $x_n = x_{(n)}$; then by Lemma 1 with $A = 0$, $B = 1$, $m = n-1$ and $a_i = x_i$ for each $i = 1, \dots, m$, we have

$$\sum_{i=1}^{n} f(x_i) = f(x_{(n)}) + \sum_{i=1}^{n-1} f(x_i) \leq f(x_{(n)}) + (n-1)f\left(\frac{1 - x_{(n)}}{n-1}\right).$$

Reapplying Lemma 1, this time with $m = n$, $a_1 = \cdots = a_{n-1} = (1 - x_{(n)})/(n-1)$ and $a_n = x_{(n)}$, we get

$$f(x_{(n)}) + (n-1)f\left(\frac{1 - x_{(n)}}{n-1}\right) \leq nf(1/n).$$

Taken together, the previous two inequalities give (3.4). As for (3.5), an application of Lemma 2 with $m = n$, $A = x_{(n)}$, $B = 1$ and $a_i = x_i$ for $i = 1, \dots, n$ gives

$$\sum_{i=1}^{n} f(x_i) \geq (n-1)f(x_{(n)}) + f(1-(n-1)x_{(n)}).$$

Reapplying the lemma with $A = 0$, $a_1 = \cdots = a_{n-1} = x_{(n)}$ and $a_n = 1-(n-1)x_{(n)}$ gives

$$(n-1)f(x_{(n)}) + f(1-(n-1)x_{(n)}) \geq (n-1)f(0) + f(1).$$

The previous two inequalities give (3.5). □

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